The Bell test for non-specialists who know about vectors

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In quantum mechanics, we consider the outcome of a single measurement as random. But is this randomness due to a lack of knowledge? In other words, is there some hidden information that actually perfectly determines the outcome of the experiment? Let us use the Bell test to find out.

1 Super basic quantum mechanics

We will consider an experiment performed on two particles with two possible states each. The quantum mechanical description of such a system can be very clearly defined in terms of a bit of linear algebra as follows.

A quantum state is represented by a vector. We will consider a system of two electrons, each with either spin up or down, so we will consider the basis

$$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}, \tag{1.1}$$

such that the state of both electrons being up is the vector

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.2}$$

and so on. If $|\cdot\rangle$ is some vector, then $|\cdot\rangle^\dagger = \langle\cdot|$ is its conjugate transpose. To measure the spin of the first electron along its spin axis (which we define as the z-axis), we define the matrix corresponding to the measurement

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.3}$$

which lets us predict the expectation value of the measurement through matrix multiplication, e.g.

$$\langle\uparrow\uparrow|Z_1|\uparrow\uparrow\rangle = 1, \qquad \langle\downarrow\uparrow|Z_1|\downarrow\uparrow\rangle = -1,. \tag{1.4}$$

This means that if we measure the z-spin of a spin up particle the expectation value is positive, and if we measure the same of a spin down particle, it is negative. The expectation value represents the average result over many experiments, which is useful for systems in a mix between different basis states. We can define the matrix representation of Z_2 which measures z-axis spin of the second electron, as well as X_1 and X_2 which measure the x-axis spin of the two electrons. These are written out at the end of the text.

Consider then a system with two particles in the state

$$|\psi\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{1.5}$$

In our basis this has the vector representation

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \tag{1.6}$$

The state $|\psi\rangle$ is the superposition of the two cases (up, down) and (down, up), each with equal probability. So from this state we can predict that measuring the spin of the first electron is equally probable to yield up or down, and upon this measurement, we can also know the spin of the other electron, which is opposite.

The big question is then whether there is some hidden information which would allow us to fully predict whether that first electron is up or down. If there is, the superposition represents our lack of knowledge, but if there is not, the superposition represents something more. We define these two cases as follows:

1. The electron spin that we measure is already determined before measurement, and this information is simply hidden from us until we measure it (**hidden information**).
2. The electrons are fundamentally described by $|\psi\rangle$, and it is impossible to predict which is up and which is down because the electrons do not even "know" themselves (**fundamental randomness**).

If case 1 was true, we could use simple probability theory to make predictions about the electrons, while if case 2 was true, doing that would conflict with experiments.

2 CHSH inequality

If case 1 is true, the quantum experiment is not fundamentally different from the following "box" experiment. Bear with me, as it needs to be somewhat contrived to be directly comparable to the quantum experiment.

We take a black and a white chess pawn and put them in two identical boxes. The pawns have definite colors, but that information is hidden. We secretly shuffle the boxes and give Alice and Bob a box each. Alice has two machines into which she can feed the box upon which they will report a result between $+1$ and -1 depending on the color of the pawn inside. She can only choose one machine per box, and they do not have to work the same way. We will label their results as A_1 and A_2, respectively. Bob has two similar machines B_1 and B_2. Let us then say that we would like to investigate the behavior of the quantity

$$F = (A_1 + A_2)B_1 + (A_1 - A_2)B_2, \tag{2.1}$$

as we repeat the box experiment many times. Each time F can take on different values depending on who gets which pawn, and depending on the rules that determine the machine outputs (these are constant between runs). But since F has the maximum value[1] $\max(F) = 2$, we know that the average of F obeys

$$\langle F\rangle \leq 2. \tag{2.2}$$

This is called the CHSH inequality.

3 The quantum experiment

Now let us consider the corresponding quantum experiment. If case 2 is true, this should be fundamentally different from the box experiment. We consider again two electrons in the state

$$|\psi\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \tag{3.1}$$

and pass one to Alice and one to Bob, who perform the measurements

$$A_1 = Z_1, \tag{3.2}$$

$$A_2 = X_1, \tag{3.3}$$

$$B_1 = -\frac{1}{\sqrt{2}}(Z_2 + X_2), \tag{3.4}$$

$$B_2 = \frac{1}{\sqrt{2}}(Z_2 - X_2), \tag{3.5}$$

[1] Each term contributes maximally when B_1 and B_2 are ± 1 with the same sign as their respective parenthesis, leading to either $F = 2A_{1/2}$ depending on sign.

where Z_1 means measuring the z-axis spin of the first electron. These measurements will all have results between $+1$ and -1. The matrix representation of F as defined in Eq. (2.1) is then

$$F = -\sqrt{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}. \tag{3.6}$$

The average $\langle F\rangle$ corresponds to the quantum mechanical expectation value, and using Eq. (1.6) we find it to be

$$\langle\psi|F|\psi\rangle = 2\sqrt{2}. \tag{3.7}$$

In other words, quantum mechanics predict that this experiment will break the CHSH inequality!

4 Consequences

At this point, it has been extremely clearly confirmed by experiments that indeed $\langle F\rangle > 2$ for the quantum system described above. So it seems the quantum mechanical description captures something which is absent from the hidden information description. The superposition does not describe a lack of knowledge about the system, but rather that the electrons fundamentally are fundamentally in the "indeterminate" state $|\psi\rangle$ until they are measured, and thus that there can exist no information with which we can predict the experiment outcome precisely. This is the Bell theorem: *no theory relying on hidden information can reproduce the results of quantum mechanics.*

There is a loophole, however: if one allows Alice and Bob to communicate with each other at faster-than-light speeds, it is possible to construct a deterministic hidden-information theory which reproduces quantum mechanics, the so-called pilot wave theory. This is generally considered a niche theory since the law of information never traveling faster than light is a fundamental result of the theory of relativity.

5 Matrix representations

For reference, the full matrix representations are

$$Z_1 = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, Z_2 = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, X_1 = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, X_2 = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{5.1}$$

The attentive reader might notice that applying the X matrices to a state flips the z spins of the state. This is a consequence of describing x spins in a z spin basis, but that is a story for another time.